q-difference intertwining operators for $U_q(sl(n))$: general setting and the case n=3

q-difference intertwining operators for $U_q(sl(n))$: general setting and the case $n = 3$

V K Dobrev†

Arnold Sommerfeld Institute for Mathematical Physics, Technical University Clausthal, Leibnizstrasse 10, 38678 Clausthal-Zellerfeld, Germany

Abstract. We construct representations $\hat{\pi}_{\bar{r}}$ of the quantum algebra $U_q(sl(n))$ labelled by $n - 1$ complex numbers r_i and acting in the space of formal power series of $n(n-1)/2$ non-commuting variables. These variables generate a flag manifold of the matrix quantum group $SL_q(n)$ which is dual to $U_q(sl(n))$. The conditions for reducibility of $\hat{\pi}_{\bar{r}}$ and the procedure for the construction of the q-difference intertwining operators are given. The representations and q-difference intertwining operators are given in the most explicit form for $n = 3$.

1. Introduction

Invariant differential equations $\mathcal{I}f = 0$ play a very important role in the description of physical symmetries—recall, e.g. the examples of Dirac and Maxwell's equations (for more examples cf e.g. [1]). It is an important and yet unsolved problem to find such equations for the setting of quantum groups, where they are expected as q-difference equations, especially in the case of non-commuting variables.

The approach to this problem used here relies on the following. In the classical situation the invariant differential operators $\mathcal{I}$ giving the equations above may be described as operators intertwining representations of complex and real semisimple Lie groups [2–5]. There are many ways to find such operators (cf e.g. [1]); however, most of these rely on constructions which are not available for quantum groups. Here we shall apply a procedure [5] which is rather algebraic and can be generalized almost straightforwardly to quantum groups. According to this procedure, one first needs to know these constructions for the complex semisimple Lie groups since the consideration of a real semisimple Lie group involves also its complexification. That is why we start here with the case of $U_q(sl(n))$ (we write $sl(n)$ instead of $sl(n, \mathbb{C})$). For the procedure one needs q-difference realizations of the representations in terms of functions of non-commuting variables. Until now such a realization of the representations and of the intertwining operators was found only for a Lorentz quantum algebra (dual to the matrix Lorentz quantum group of [6] in [7]. The construction in [7] (also applying the procedure of [5]) involves two q-commuting variables $\eta\bar{\eta} = q\bar{\eta}\eta$ and uses the complexification $U_q(sl(2)) \otimes U_q(sl(2))$ of the Lorentz quantum algebra.

In the present paper, following the above-mentioned procedure, we construct representations $\hat{\pi}_{\bar{r}}$ of $U_q(sl(n))$ labelled by $n - 1$ complex numbers $\bar{r} = \{r_1, \ldots, r_{n-1}\}$

and acting in the spaces of formal power series of $n(n - 1)/2$ non-commuting (for $n > 2$) variables Y_{ij}, $1 \leqslant j < i \leqslant n$. These variables generate a flag manifold of the matrix quantum group $SL_q(n)$, which is dual to $U_q(sl(n))$. For generic $r_i \in \mathbb{C}$ the representations $\hat{\pi}_{\vec{r}}$ are irreducible. We give the values of r_i when the representations $\hat{\pi}_{\vec{r}}$ are reducible. It is in the latter cases that there arise various partial equivalences among these representations. These partial equivalences are realized by q-difference intertwining operators for which we give a canonical derivation following [5]. For $q = 1$ these operators become the invariant differential operators mentioned above. We should also note that our considerations below are in general for $n \geqslant 2$, though the case $n = 2$, while being done first as a toy model [8], is not interesting from the non-commutative point of view since it involves functions of one variable; furthermore, the representations and the only possible q-difference intertwining operator are known for $U_q(sl(2))$ (though derived by a different method) [9].

The paper is organized as follows. In section 2 we recall the matrix quantum group $GL_q(n)$ and its dual quantum algebra $\mathcal{U}_g$. In section 3 we give the explicit construction of representations of $\mathcal{U}_g$ and its semisimple part $U_q(sl(n))$. In section 4 we give the reducibility conditions for these representations and the procedure for the construction of the q-difference intertwining operators. In section 5 we consider the case $n = 3$ in more detail.

2. The matrix quantum group

Let us consider an $n \times n$ quantum matrix M with non-commuting matrix elements a_{ij}, $1 \leqslant i, j \leqslant n$. The matrix quantum group $\mathcal{A}_g = GL_q(n)$, $q \in \mathbb{C}$, is generated by the matrix elements a_{ij} with the following commutation relations [10] ($\lambda = q - q^{-1}$):

$$
\begin{align}
a_{ij}a_{i\ell} &= q^{-1}a_{i\ell}a_{ij} & \text{for} \quad j < \ell & \tag{1a} \\
a_{kj} &= q^{-1}a_{kj}a_{ij} & \text{for} \quad i < k & \tag{1b} \\
a_{i\ell}a_{kj} &= a_{kj}a_{i\ell} & \text{for} \quad i < k, \quad j < \ell & \tag{1c} \\
a_{k\ell}a_{ij} - a_{ij}a_{k\ell} &= \lambda a_{i\ell}a_{kj} & \text{for} \quad i < k, \quad j < \ell. & \tag{1d}
\end{align}
$$

Considered as a bialgebra, it has the following comultiplication δ_A and counit ε_A:

$$
\delta_A(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj} \qquad \varepsilon_A(a_{ij}) = \delta_{ij}. \tag{2}
$$

This algebra has a determinant D given by [10],

$$
D = \sum_{\rho \in S_n} \epsilon(\rho)a_{1,\rho(1)} \ldots a_{n,\rho(n)} = \sum_{\rho \in S_n} \epsilon(\rho)a_{\rho(1),1} \ldots a_{\rho(n),n} \tag{3}
$$

where summations are over all permutations ρ of $\{1, \ldots, n\}$ and the quantum signature is

$$
\epsilon(\rho) = \prod_{\substack{j<k \\ \rho(j)>\rho(k)}} (-q^{-1}). \tag{4}
$$

The determinant obeys [10]:

$$
\delta_A(D) = D \otimes D \qquad \varepsilon_A(D) = 1. \tag{5}
$$

The determinant is central, i.e. it commutes with the elements a_{ik} [10]:

$$a_{ik}D = Da_{ik}. \tag{6}$$

Further, if $D \neq 0$, one extends the algebra by an element D^{-1} which obeys [10]:

$$DD^{-1} = D^{-1}D = 1_A. \tag{7}$$

Next one defines the left and right quantum cofactor matrix A_{ij} [10]:

$$
\begin{aligned}
A_{ij} &= \sum_{\rho(i)=j} \frac{\epsilon(\rho \circ \sigma_i)}{\epsilon(\sigma_i)} a_{1,\rho(1)} \dots \widehat{a}_{ij} \dots a_{n,\rho(n)} \\
&= \sum_{\rho(j)=i} \frac{\epsilon(\rho \circ \sigma_j')}{\epsilon(\sigma_j')} a_{\rho(1),1} \dots \widehat{a}_{ij} \dots a_{\rho(n),n}
\end{aligned} \tag{8}
$$

where σ_i and σ_j' denote the cyclic permutations,

$$\sigma_i = \{i, \dots, 1\} \qquad \sigma_j' = \{j, \dots, n\} \tag{9}$$

and the notation $\hat{x}$ indicates that x is to be omited. Now one can show that [10]:

$$\sum_j a_{ij} A_{\ell j} = \sum_j A_{ji} a_{j\ell} = \delta_{i\ell} D \tag{10}$$

and obtain the left and right inverse [10]:

$$M^{-1} = D^{-1}A = AD^{-1}. \tag{11}$$

Thus, one can introduce the antipode in $GL_q(n)$ [10]:

$$\gamma_A(a_{ij}) = D^{-1}A_{ji} = A_{ji}D^{-1}. \tag{12}$$

Next we introduce a basis of $GL_q(n)$ which consists of monomials

$$f = (a_{21})^{p_{21}} \dots (a_{n,n-1})^{p_{n,n-1}} (a_{11})^{\ell_1} \dots (a_{nn})^{\ell_n} (a_{n-1,n})^{n_{n-1,n}} \dots (a_{12})^{n_{12}} = f_{\bar{\ell},\bar{p},\bar{n}} \tag{13}$$

where $\bar{\ell}, \bar{p}, \bar{n}$ denote the sets $\{\ell_i\}, \{p_{ij}\}, \{n_{ij}\}$, respectively, $\ell_i, p_{ij}, n_{ij} \in \mathbb{Z}_+$ and we have used the so-called normal ordering of the elements a_{ij}. Namely, we first put the elements a_{ij} with $i > j$ in lexicographic order, i.e. if $i < k$ then a_{ij} ($i > j$) is before $a_{k\ell}$ ($k > \ell$) and a_{ti} ($t > i$) is before a_{tk} ($t > k$); then we put the elements a_{ii}; finally we put the elements a_{ij} with $i < j$ in antilexicographic order, i.e. if $i > k$ then a_{ij} ($i < j$) is before $a_{k\ell}$ ($k < \ell$) and a_{ti} ($t < i$) is before a_{tk} ($t < k$). Note that the basis (13) also includes the unit element $1_{\mathcal{A}_g}$ of $\mathcal{A}_g$ when all $\{\ell_i\}, \{p_{ij}\}, \{n_{ij}\}$ are equal to zero, i.e.

$$f_{\bar{0},\bar{0},\bar{0}} = 1_{\mathcal{A}_g}. \tag{14}$$

We need the dual algebra of $GL_q(n)$. This is the algebra $\mathcal{U}_g = U_q(sl(n)) \otimes U_q(\mathcal{Z})$, where $U_q(\mathcal{Z})$ is central in $\mathcal{U}_g$ [11]. Let us denote the Chevalley generators of $sl(n)$ by H_i,

$X_i^\pm$, $i = 1, \ldots, n - 1$. Then we choose $k_i = q^{H_i/2}$, $k_i^{-1} = q^{-H_i/2}$, $X_i^\pm$, $i = 1, \ldots, n - 1$ for the 'Chevalley' generators of $\mathcal{U} = U_q(sl(n))$, with the *algebra* relations

$$k_i k_j = k_j k_i \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1_{\mathcal{U}_g} \qquad k_i X_j^\pm = q^{\pm c_{ij}} X_j^\pm k_i \tag{15a}$$

$$[X_i^+, X_j^-] = \delta_{ij} \left(k_i^2 - k_i^{-2} \right) / \lambda \tag{15b}$$

$$\left(X_i^\pm \right)^2 X_j^\pm - [2]_q X_i^\pm X_j^\pm X_i^\pm + X_j^\pm \left(X_i^\pm \right)^2 = 0 \qquad |i - j| = 1 \tag{15c}$$

$$[X_i^\pm, X_j^\pm] = 0 \qquad |i - j| \neq 1 \tag{15d}$$

where c_{ij} is the Cartan matrix of $sl(n)$, and the *coalgebra* relations

$$\delta_{\mathcal{U}}(k_i^\pm) = k_i^\pm \otimes k_i^\pm \tag{16a}$$

$$\delta_{\mathcal{U}}(X_i^\pm) = X_i^\pm \otimes k_i + k_i^{-1} \otimes X_i^\pm \tag{16b}$$

$$\varepsilon_{\mathcal{U}}(k_i^\pm) = 1 \qquad \varepsilon_{\mathcal{U}}(X_i^\pm) = 0 \tag{16c}$$

$$\gamma_{\mathcal{U}}(k_i) = k_i^{-1} \qquad \gamma_{\mathcal{U}}(X_i^\pm) = -q^{\pm 1} X_i^\pm \tag{16d}$$

where $k_i^+ = k_i$, $k_i^- = k_i^{-1}$. Further, we denote the generator of $\mathcal{Z}$ by H and the generators of $U_q(\mathcal{Z})$ by $k = q^{H/2}$, $k^{-1} = q^{-H/2}$, $kk^{-1} = k^{-1}k = 1_{\mathcal{U}_g}$. The generators k, k^{-1} commute with the generators of $\mathcal{U}$, and their coalgebra relations are as those of any k_i. From now on we shall give most formulae only for the generators k_i, $X_i^\pm$, k, since the analogous formulae for k_i^{-1}, k^{-1} follow trivially from those for k_i, k, respectively.

The bilinear form giving the duality between $\mathcal{U}_g$ and $\mathcal{A}_g$ is given by [11]

$$\langle k_i, a_{j\ell} \rangle = \delta_{j\ell} q^{(\delta_{ij} - \delta_{i,j+1})/2} \tag{17a}$$

$$\langle X_i^+, a_{j\ell} \rangle = \delta_{j+1,\ell} \delta_{ij} \tag{17b}$$

$$\langle X_i^-, a_{j\ell} \rangle = \delta_{j-1,\ell} \delta_{i\ell} \tag{17c}$$

$$\langle k, a_{j\ell} \rangle = \delta_{j\ell} q^{1/2}. \tag{17d}$$

The pairing between arbitrary elements of $\mathcal{U}_g$ and f follows then from the properties of the duality pairing. All this is given in [11] and is not reproduced here since we shall not need these formulae. The pairing (17) is typically supplemented with

$$\langle y, 1_{\mathcal{A}_g} \rangle = \varepsilon_{\mathcal{U}_g}(y). \tag{18}$$

It is well known that the pairing provides the fundamental representation of $\mathcal{U}_g$:

$$F(y)_{j\ell} = \langle y, a_{j\ell} \rangle \qquad y = k_i, X_i^\pm, k. \tag{19}$$

Of course, $F(k) = q^{1/2} I_n$, where I_n is the unit $n \times n$ matrix.

3. Representations of $\mathcal{U}_g$ and $\mathcal{U}$

We begin by defining *two actions* of the dual algebra $\mathcal{U}_g$ on the basis (13) of $\mathcal{A}_g$.

First we introduce the *left regular representation* of $\mathcal{U}_g$ which in the $q = 1$ case is the infinitesimal version of

$$\pi(Y) M = Y^{-1} M \qquad Y, M \in GL(n). \tag{20}$$

Explicitly, we define the action of $\mathcal{U}_g$ as (cf (19))

$$\pi(y)a_{i\ell} \doteq \left(F\left(y^{-1}\right)M\right)_{i\ell} = \sum_j F\left(y^{-1}\right)_{ij} a_{j\ell} = \sum_j \langle y^{-1}, a_{ij}\rangle a_{j\ell} \tag{21}$$

where y denotes the generators of $\mathcal{U}_g$ and y^{-1} is symbolic notation, the possible pairs being given explicitly by

$$(y, y^{-1}) = (k_i, k_i^{-1}) \qquad (X_i^{\pm}, -X_i^{\pm}) \qquad (k, k^{-1}). \tag{22}$$

From (21) we find the explicit action of the generators of $\mathcal{U}_g$:

$$\pi(k_i)a_{j\ell} = q^{(\delta_{i+1,j}-\delta_{ij})/2}a_{j\ell} \tag{23a}$$

$$\pi(X_i^+)a_{j\ell} = -\delta_{ij}a_{j+1\ell} \tag{23b}$$

$$\pi(X_i^-)a_{j\ell} = -\delta_{i+1,j}a_{j-1\ell} \tag{23c}$$

$$\pi(k)a_{j\ell} = q^{-1/2}a_{j\ell}. \tag{23d}$$

The above is supplemented with the following action on the unit element of $\mathcal{A}_g$:

$$\pi(k_i)1_{\mathcal{A}_g} = 1_{\mathcal{A}_g} \qquad \pi(X_i^{\pm})1_{\mathcal{A}_g} = 0 \qquad \pi(k)1_{\mathcal{A}_g} = 1_{\mathcal{A}_g}. \tag{24}$$

In order to derive the action of $\pi(y)$ on arbitrary elements of the basis (13), we use the twisted derivation rule consistent with the coproduct and the representation structure, namely, we take: $\pi(y)\varphi\psi = \pi(\delta'_{\mathcal{U}_g}(y))(\varphi \otimes \psi)$, where $\delta'_{\mathcal{U}_g} = \sigma \circ \delta_{\mathcal{U}_g}$ is the opposite coproduct (σ is the permutation operator). Thus, we have

$$\pi(k_i)\varphi\psi = \pi(k_i)\varphi \cdot \pi(k_i)\psi \tag{25a}$$

$$\pi(X_i^{\pm})\varphi\psi = \pi(X_i^{\pm})\varphi \cdot \pi(k_i^{-1})\psi + \pi(k_i)\varphi \cdot \pi(X_i^{\pm})\psi \tag{25b}$$

$$\pi(k)\varphi\psi = \pi(k)\varphi \cdot \pi(k)\psi . \tag{25c}$$

From now on we suppose that q is not a non-trivial root of unity. Applying the above rules one obtains

$$\pi(k_i)(a_{j\ell})^n = q^{n(\delta_{i+1,j}-\delta_{ij})/2}(a_{j\ell})^n \tag{26a}$$

$$\pi(X_i^+)(a_{j\ell})^n = -\delta_{ij}c_n(a_{j\ell})^{n-1}a_{j+1\ell} \tag{26b}$$

$$\pi(X_i^-)(a_{j\ell})^n = -\delta_{i+1,j}c_na_{j-1\ell}(a_{j\ell})^{n-1} \tag{26c}$$

$$\pi(k)(a_{j\ell})^n = q^{-n/2}(a_{j\ell})^n \tag{26d}$$

where

$$c_n = q^{(n-1)/2}[n]_q \qquad [n]_q = (q^n - q^{-n})/\lambda. \tag{27}$$

Note that (24) and (23) are partial cases of (26) for $n = 0$ and $n = 1$, respectively (cf (14)).

Analogously, we introduce the *right action* (see also [12]) which in the classical case is the infinitesimal counterpart of

$$\pi_R(Y)M = MY \qquad Y, M \in GL(n). \tag{28}$$

Thus, we define the right action of $\mathcal{U}_g$ as (cf (19))

$$\pi_R(y)a_{i\ell} = (MF(y))_{i\ell} = \sum_j a_{ij}F(y)_{j\ell} = \sum_j a_{ij}\langle y, a_{j\ell}\rangle \tag{29}$$

where y denotes the generators of $\mathcal{U}_g$.

From (29) we find the explicit right action of the generators of $\mathcal{U}_g$:

$$\pi_R(k_i)a_{j\ell} = q^{(\delta_{i\ell}-\delta_{i+1,\ell})/2}a_{j\ell} \tag{30a}$$

$$\pi_R(X_i^+)a_{j\ell} = \delta_{i+1,\ell}a_{j,\ell-1} \tag{30b}$$

$$\pi_R(X_i^-)a_{j\ell} = \delta_{i\ell}a_{j,\ell+1} \tag{30c}$$

$$\pi_R(k)a_{j\ell} = q^{1/2}a_{j\ell} \tag{30d}$$

supplemented by the right action on the unit element:

$$\pi_R(k_i)1_{A_g} = 1_{A_g}, \qquad \pi_R(X_i^\pm)1_{A_g} = 0 \qquad \pi_R(k)1_{A_g} = 1_{A_g}. \tag{31}$$

The twisted derivation rule is now given by $\pi_R(y)\varphi\psi = \pi_R(\delta_{\mathcal{U}_g}(y))(\varphi \otimes \psi)$, i.e.

$$\pi_R(k_i)\varphi\psi = \pi_R(k_i)\varphi \cdot \pi_R(k_i)\psi \tag{32a}$$

$$\pi_R(X_i^\pm)\varphi\psi = \pi_R(X_i^\pm)\varphi \cdot \pi_R(k_i)\psi + \pi_R(k_i^{-1})\varphi \cdot \pi_R(X_i^\pm)\psi \tag{32b}$$

$$\pi_R(k)\varphi\psi = \pi_R(k)\varphi \cdot \pi_R(k)\psi. \tag{32c}$$

Using this, we find

$$\pi_R(k_i)(a_{j\ell})^n = q^{n(\delta_{i\ell}-\delta_{i+1,\ell})/2}(a_{j\ell})^n \tag{33a}$$

$$\pi_R(X_i^+)(a_{j\ell})^n = \delta_{i+1,\ell}c_n a_{j,\ell-1}(a_{j\ell})^{n-1} \tag{33b}$$

$$\pi_R(X_i^-)(a_{j\ell})^n = \delta_{i\ell}c_n(a_{j\ell})^{n-1}a_{j,\ell+1} \tag{33c}$$

$$\pi_R(k)(a_{j\ell})^n = q^{n/2}(a_{j\ell})^n. \tag{33d}$$

Let us now introduce the elements φ as formal power series of the basis (13):

$$\varphi = \sum_{\bar{\ell},\bar{m},\bar{n}\in\mathbb{Z}_+} \mu_{\bar{\ell},\bar{m},\bar{n}}(a_{21})^{m_{21}}\ldots(a_{n,n-1})^{m_{n,n-1}}(a_{11})^{\ell_1}\ldots(a_{nn})^{\ell_n}$$

$$\times (a_{n-1,n})^{n_{n-1,n}}\ldots(a_{12})^{n_{12}}. \tag{34}$$

By (26) and (33) we have defined the left and the right actions of $\mathcal{U}_g$ on φ. As in the classical case, the left and right actions commute, and as in [5] we shall use the right action to reduce the left regular representation (which is highly reducible). In particular, we would like the right action to mimic some properties of a highest-weight module, i.e. the annihilation by the raising generators X_i^+ and scalar action by the (exponents of the) Cartan operators k_i, k. In the classical case these properties are also called right covariance [5]. However, first we have to execute a change of basis using the q-analogue of the classical Gauss decomposition. For this we have to suppose that the principal minor determinants of M,

$$D_m = \sum_{\rho\in S_m} \epsilon(\rho)a_{1,\rho(1)}\ldots a_{m,\rho(m)} = \sum_{\rho\in S_m} \epsilon(\rho)a_{\rho(1),1}\ldots a_{\rho(m),m} \qquad m \leqslant n \tag{35}$$

are invertible; note that $D_n = D$, $D_{n-1} = A_{nn}$. Thus, using (10) for $i = \ell = n$ we can express, e.g. a_{nn} in terms of other elements:

$$a_{nn} = \left(D - \sum_{j<n} a_{nj} A_{nj} \right) D_{n-1}^{-1} = D_{n-1}^{-1} \left(D - \sum_{j<n} A_{jn} a_{jn} \right). \tag{36}$$

Further, for the ordered sets $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$, let ξ_J^I be the r-minor determinant with respect to rows I and columns J, such that

$$\xi_J^I = \sum_{\rho \in S_r} \epsilon(\rho) a_{i_{\rho(1)} j_1} \cdots a_{i_{\rho(r)} j_r}. \tag{37}$$

Note that $\xi_{1 \cdots i}^{1 \cdots i} = D_i$. Then one has [13] $(i, j, \ell = 1, \ldots, n)$

$$a_{i\ell} = \sum_j B_{ij} Z_{j\ell} \qquad B_{i\ell} = \xi_{1 \cdots \ell}^{1 \cdots \ell-1\,i} D_{\ell-1}^{-1} \qquad Z_{i\ell} = D_i^{-1} \xi_{1 \cdots i-1\,\ell}^{1 \cdots i} \tag{38}$$

$B_{i\ell} = 0$ for $i < \ell$, $Z_{i\ell} = 0$ for $i > \ell$, (which follows from the obvious extension of (37) to the case when I (respectively, J) is not ordered). Then Z_{ij}, $i < j$, may be regarded as a q-analogue of local coordinates of the flag manifold $B \backslash GL(n)$.

For our purposes we need a refinement of this decomposition:

$$B_{i\ell} = Y_{i\ell} D_{\ell\ell} \qquad Y_{i\ell} = \xi_{1 \cdots \ell}^{1 \cdots \ell-1\,i} D_\ell^{-1} \qquad D_{\ell\ell} = D_\ell D_{\ell-1}^{-1} \qquad (D_0 \equiv 1_{A_g}) \tag{39}$$

where $Y_{j\ell}$, $j > \ell$, may be regarded as a q-analogue of local coordinates of the flag manifold $GL(n)/DZ$.

Clearly, we can replace the basis (13) of A_g with a basis in terms of $Y_{i\ell}$, $i > \ell$, D_ℓ, $Z_{i\ell}$, $i < \ell$. (Note that $Y_{ii} = Z_{ii} = 1_{A_g}$.) We could have used also $D_{\ell\ell}$ instead of D_ℓ, but this choice is more convenient since we shall impose $D_n = D = 1_{A_g}$ below. Thus, we consider formal power series:

$$\varphi = \sum_{\substack{\bar{m}, \bar{n} \in \mathbb{Z}_+ \\ i \in \mathbb{Z}}} \mu'_{\bar{\ell}, \bar{m}, \bar{n}} (Y_{21})^{m_{21}} \ldots (Y_{n,n-1})^{m_{n,n-1}} (D_1)^{\ell_1} \ldots (D_n)^{\ell_n} (Z_{n-1,n})^{n_{n-1,n}} \ldots (Z_{12})^{n_{12}}. \tag{40}$$

Now, let us impose right covariance [5] with respect to X_i^+, i.e. we require

$$\pi_R(X_i^+)\varphi = 0. \tag{41}$$

First we notice that

$$\pi_R(X_i^+)\xi_J^I = 0 \qquad \text{for } J = \{1, \ldots, j\}, \ \forall I \tag{42}$$

from which we have

$$\pi_R(X_i^+)D_j = 0 \qquad \pi_R(X_i^+)Y_{j\ell} = 0. \tag{43}$$

On the other hand, $\pi_R(X_i^+)$ acts non-trivially on $Z_{j\ell}$. Thus, (41) simply means that our functions φ do not depend on $Z_{j\ell}$. The functions obeying (41) are therefore

$$\varphi = \sum_{\bar{\ell} \in \mathbb{Z}, \bar{m} \in \mathbb{Z}_+} \mu_{\bar{\ell}, \bar{m}} (Y_{21})^{m_{21}} \ldots (Y_{n,n-1})^{m_{n,n-1}} (D_1)^{\ell_1} \ldots (D_n)^{\ell_n}. \tag{44}$$

Next, we impose right covariance with respect to k_i and k,

$$\pi_R(k_i)\varphi = q^{r_i/2}\varphi \tag{45a}$$

$$\pi_R(k)\varphi = q^{\hat{r}/2}\varphi \tag{45b}$$

where r_i and $\hat{r}$ are parameters to be specified below. On the other hand, using $(32a, c)$ and $(33a, c)$, we have

$$\pi_R(k_i)\xi_J^I = q^{\delta_{ij}/2}\xi_J^I \qquad \pi_R(k)\xi_J^I = q^{j/2}\xi_J^I \qquad \text{for } J = \{1, \dots, j\}, \ \forall I \tag{46}$$

from which we have

$$\pi_R(k_i)D_j = q^{\delta_{ij}/2}D_j \qquad \pi_R(k)D_j = q^{j/2}D_j \tag{47a}$$

$$\pi_R(k_i)Y_{j\ell} = Y_{j\ell} \qquad \pi_R(k)Y_{j\ell} = Y_{j\ell} \tag{47b}$$

and thus we obtain

$$\pi_R(k_i)\varphi = q^{\ell_i/2}\varphi \tag{48a}$$

$$\pi_R(k)\varphi = q^{\sum_{j=1}^{n} j\ell_j/2}\varphi. \tag{48b}$$

Comparing right covariance conditions (45) with the direct calculations [48] we obtain $\ell_i = r_i$ for $i < n$, $\sum_{j=1}^{n} j\ell_j = \hat{r}$. This means that $r_i, \hat{r} \in \mathbb{Z}$ and that there is no summation in ℓ_i; also, $\ell_n = (\hat{r} - \sum_{i=1}^{n-1} ir_i)/n$.

Thus, the reduced functions obeying (41) and (45) are

$$\varphi = \sum_{\tilde{m}\in\mathbb{Z}_+} \mu_{\tilde{m}}(Y_{21})^{m_{21}} \dots (Y_{n,n-1})^{m_{n,n-1}}(D_1)^{r_1} \dots (D_{n-1})^{r_{n-1}}(D_n)^{\hat{\ell}} \tag{49}$$

where $\hat{\ell} = (\hat{r} - \sum_{i=1}^{n-1} ir_i)/n$.

Next we would like to derive the $\mathcal{U}_g$ action π on φ. First, we notice that $\mathcal{U}$ acts trivially on $D_n = D$:

$$\pi(X_i^{\pm})D = 0 \qquad \pi(k_i)D = D. \tag{50}$$

Then we note that

$$\pi(k)D_j = q^{-j/2}D_j \qquad \pi(k)Y_{j\ell} = Y_{j\ell} \tag{51}$$

which implies

$$\pi(k)\varphi = q^{-\hat{r}/2}\varphi. \tag{52}$$

Thus, the action of $\mathcal{U}$ involves only the parameters r_i, $i < n$, while the action of $U_q(\mathcal{Z})$ involves only the parameter $\hat{r}$. We can therefore, also from the representation theory standpoint, restrict consistently to the matrix quantum group $SL_q(n)$, i.e. we set

$$D = D^{-1} = 1_{\mathcal{A}_t}. \tag{53}$$

Then the dual algebra is $\mathcal{U} = U_q(sl(n))$. This is justified as in the $q = 1$ case [5], since for our considerations only the semisimple part of the algebra is important. (This would not be

possible for the multiparameter deformation of $GL(n)$ [14, 15], since D is not central there. Nevertheless, we expect most of the essential features of our approach to be preserved since the dual algebra can be transformed as a commutation algebra to the one-parameter $\mathcal{U}_g$, with the extra parameters entering only the co-algebra structure [11].)

The reduced functions for the $\mathcal{U}$ action are, therefore,

$$\tilde{\varphi}(\bar{Y}, \bar{D}) = \sum_{\bar{m} \in \mathbb{Z}_+} \mu_{\bar{m}}(Y_{21})^{m_{21}} \dots (Y_{n,n-1})^{m_{n,n-1}} (D_1)^{r_1} \dots (D_{n-1})^{r_{n-1}} \tag{54a}$$

$$= \hat{\varphi}(\bar{Y})(D_1)^{r_1} \dots (D_{n-1})^{r_{n-1}} \tag{54b}$$

where $\bar{Y}$, $\bar{D}$ denote the variables Y_{il}, $i > \ell$, D_i, $i < n$. Next we calculate

$$\pi(k_i)D_j = q^{-\delta_{ij}/2}D_j \tag{55a}$$

$$\pi(X_i^+)D_j = -\delta_{ij}Y_{j+1,j}D_j \tag{55b}$$

$$\pi(X_i^-)D_j = 0 \tag{55c}$$

$$\pi(k_i)Y_{j\ell} = q^{\frac{1}{2}(\delta_{i+1,j} - \delta_{ij} - \delta_{i+1,\ell} + \delta_{i\ell})}Y_{j\ell} \tag{56a}$$

$$\pi(X_i^+)Y_{j\ell} = -\delta_{ij}Y_{j+1,\ell} + \delta_{i\ell}q^{1-\delta_{i,\ell+1}/2}Y_{\ell+1,\ell}Y_{j\ell} + \delta_{i+1,\ell}\left(q^{-1}Y_{j,\ell-1} - Y_{\ell,\ell-1}Y_{j\ell}\right) \tag{56b}$$

$$\pi(X_i^-)Y_{j\ell} = -\delta_{i+1,j}q^{-\delta_{i\ell}/2}Y_{j-1,\ell}. \tag{56c}$$

These results have the important consequence that the degrees of the variables D_j are not changed by the action of $\mathcal{U}$. For example, the parameters r_i do characterize the action of $\mathcal{U}$, i.e. we have obtained representations of $\mathcal{U}$. We shall denote by $C_{\bar{r}}$ the representation space of the functions in (54) which have covariance properties (41), (45a); and we denote the representation acting in $C_{\bar{r}}$ by $\tilde{\pi}_{\bar{r}}$ (a renormalization of the explicit formulae may be done here to simplify things). To obtain this representation more explicitly one just applies (55) and (56) to the basis in (54) using (25). In particular, we have

$$\pi(k_i)(D_j)^n = q^{-n\delta_{ij}/2}(D_j)^n \qquad n \in \mathbb{Z} \tag{57a}$$

$$\pi(X_i^+)(D_j)^n = -\delta_{ij}\bar{c}_n Y_{j+1,j}(D_j)^n \qquad n \in \mathbb{Z} \tag{57b}$$

$$\pi(X_i^-)(D_j)^n = 0 \qquad n \in \mathbb{Z} \tag{57c}$$

$$\pi(k_i)(Y_{j\ell})^n = q^{\frac{n}{2}(\delta_{i+1,j} - \delta_{ij} - \delta_{i+1,\ell} + \delta_{i\ell})}(Y_{j\ell})^n \qquad n \in \mathbb{Z}_+ \tag{58a}$$

$$\pi(X_i^+)(Y_{j\ell})^n = -\delta_{ij}c_n(Y_{j\ell})^{n-1}Y_{j+1,\ell}$$
$$+ \delta_{i\ell}q^{1-n\delta_{i,\ell+1}/2}c_n Y_{\ell+1,\ell}(Y_{j\ell})^n$$
$$+ \delta_{i+1,\ell}\bar{c}_n\left(q^{-1}Y_{j,\ell-1}(Y_{j\ell})^{n-1} - Y_{\ell,\ell-1}Y_{j\ell}^n\right) \qquad n \in \mathbb{Z}_+ \tag{58b}$$

$$\pi(X_i^-)(Y_{j\ell})^n = -\delta_{i+1,j}q^{-\delta_{i,\ell+1}n/2}c_n Y_{j-1,\ell}(Y_{j\ell})^{n-1} \qquad n \in \mathbb{Z}_+ \tag{58c}$$

where

$$\bar{c}_n = q^{(1-n)/2}[n]_q. \tag{59}$$

Further, since the action of $\mathcal{U}$ is not affecting the degrees of D_i, we introduce (as in [5]) the restricted functions $\hat{\varphi}(\bar{Y})$ by the formula which is prompted in (54b):

$$\hat{\varphi}(\bar{Y}) \equiv \left(\mathcal{A}\tilde{\varphi}\right)(\bar{Y}) \doteq \tilde{\varphi}(\bar{Y}, D_1 = \dots = D_{n-1} = 1_{A_g}). \tag{60}$$

We denote the representation space of $\hat{\varphi}(\bar{Y})$ by $\hat{C}_{\bar{r}}$ and the representation acting in $\hat{C}_{\bar{r}}$ by $\hat{\pi}_{\bar{r}}$. Thus, the operator $\mathcal{A}$ acts from $C_{\bar{r}}$ to $\hat{C}_{\bar{r}}$. The properties of $\hat{C}_{\bar{r}}$ follow from the intertwining requirement for $\mathcal{A}$ [5]:

$$\hat{\pi}_{\bar{r}}\mathcal{A} = \mathcal{A}\tilde{\pi}_{\bar{r}}. \tag{61}$$

4. Reducibilty and q-difference intertwining operators

We have defined the representations $\hat{\pi}_{\bar{r}}$ for $r_i \in \mathbb{Z}$. Note, however, that we can consider the restricted functions $\hat{\varphi}(\bar{Y})$ for arbitrary complex r_i. We shall make these extension from now on, since this gives the same set of representations for $U_q(sl(n))$ as in the case $q = 1$.

Now we make some statements which are true in the classical case [5] and will be illustrated below. For any i, j, such that $1 \leqslant i \leqslant j \leqslant n-1$, we define

$$m_{ij} \equiv r_i + \cdots + r_j + j - i + 1. \tag{62}$$

Note that $m_i = m_{ii} = r_i + 1$, $m_{ij} = m_i + \cdots + m_j$. Note also that the possible choices of i, j are in one-to-one correspondence with the positive roots $\alpha = \alpha_{ij} = \alpha_i + \cdots + \alpha_j$ of the root system of $sl(n)$, the cases $i = j = 1 \ldots, n-1$ enumerating the simple roots $\alpha_i = \alpha_{ii}$. In general, $m_{ij} \in \mathbb{C}$ for the representations $\hat{\pi}_{\bar{r}}$, while $m_{ij} \in \mathbb{Z}$ for the representations $\pi_{\bar{r}}$. If $m_{ij} \notin \mathbb{N}$ for all possible i, j the representations $\hat{\pi}_{\bar{r}}$, $\pi_{\bar{r}}$ are irreducible. If $m_{ij} \in \mathbb{N}$ for some i, j, the representations $\hat{\pi}_{\bar{r}}$, $\pi_{\bar{r}}$ are reducible. The corresponding irreducible subrepresentations are still infinite-dimensional unless $m_i \in \mathbb{N}$ for all $i = 1, \ldots, n-1$. The representation spaces of the irreducible subrepresentations are invariant irreducible subspaces of our representation spaces. These invariant subspaces are spanned by functions depending on all variables $Y_{j\ell}$, except when we have $m_s = m_{s+1} = \cdots = m_{n-1} = 1$ for some $s \in \mathbb{N}$, $1 \leqslant s \leqslant n-1$. In the latter case these functions depend only on the $(s-1)(2n-s)/2$ variables $Y_{j\ell}$ with $\ell < s$, (the unrestricted subrepresentation functions still depend on D_ℓ with $\ell < s$). In particular, for $s = 2$ the restricted subrepresentation functions depend only on $n-1$ variables Y_{j1}. The latter situation is also relatively simple in the q case since these variables are q-commuting: $Y_{j1}Y_{k1} = qY_{k1}Y_{j1}$, $j > k$. (For $s = 1$ the irreducible subrepresentation is one dimensional and hence independence of any variables.)

Furthermore, for $m_{ij} \in \mathbb{N}$ the representation $\hat{\pi}_{\bar{r}}$, $\pi_{\bar{r}}$, is partially equivalent to the representation $\hat{\pi}_{\bar{r}'}$, $\pi_{\bar{r}'}$, respectively, with $m'_\ell = r'_\ell + 1$ being explicitly given by [5]

$$m'_\ell = \begin{cases} m_\ell & \text{for } \ell \neq i-1, i, j, j+1 \\ m_{\ell j} & \text{for } \ell = i-1 \\ -m_{\ell+1,j} & \text{for } \ell = i < j \\ -m_{i,\ell-1} & \text{for } \ell = j > i \\ -m_\ell & \text{for } \ell = i = j \\ m_{i\ell} & \text{for } \ell = j+1 \,. \end{cases} \tag{63}$$

These partial equivalences are realized by intertwining operators:

$$\mathcal{I}_{ij} : C_{\bar{r}} \longrightarrow C_{\bar{r}'} \qquad m_{ij} \in \mathbb{N} \tag{64a}$$

$$I_{ij} : \hat{C}_{\bar{r}} \longrightarrow \hat{C}_{\bar{r}'} \qquad m_{ij} \in \mathbb{N} \tag{64b}$$

i.e. one has

$$\mathcal{I}_{ij} \circ \pi_{\bar{r}} = \pi_{\bar{r}'} \circ \mathcal{I}_{ij} \qquad m_{ij} \in \mathbb{N} \tag{65a}$$

$$I_{ij} \circ \hat{\pi}_{\bar{r}} = \hat{\pi}_{\bar{r}'} \circ I_{ij} \qquad m_{ij} \in \mathbb{N}. \tag{65b}$$

The invariant irreducible subspace of $\hat{\pi}_{\bar{r}}$ (respectively, $\pi_{\bar{r}}$) discussed above is the intersection of the kernels of all intertwining operators acting from $\hat{\pi}_{\bar{r}}$ (respectively, $\pi_{\bar{r}}$). When all $m_i \in$

$\mathbb{N}$ the invariant subspace is finite-dimensional with dimension $\prod_{1\leqslant i\leqslant j\leqslant n-1} m_{ij}/\prod_{i=1}^{n-1} t!$, and all finite-dimensional irreps of $U_q(sl(n))$ can be obtained in this way.

We now present a canonical procedure for the derivation of these intertwining operators following the $q = 1$ procedure of [5]. By this procedure one should take as intertwiners (up to non-zero multiplicative constants)

$$\mathcal{I}_{ij}^m = \mathcal{P}_{ij}^m \left(\pi_R(X_i^-), \ldots, \pi_R(X_j^-) \right) \qquad m = m_{ij} \in \mathbb{N} \tag{66a}$$

$$I_{ij}^m = \mathcal{P}_{ij}^m \left(\hat{\pi}_R(X_i^-), \ldots, \hat{\pi}_R(X_j^-) \right) \qquad m = m_{ij} \in \mathbb{N} \tag{66b}$$

where $\mathcal{P}_{ij}^m$ is a homogeneous polynomial in each of its $(j - i + 1)$ variables of degree m. This polynomial gives a singular vector v_{ij} in a Verma module $V^{\Lambda(\bar{r})}$ with the highest weight $\Lambda(\bar{r})$ determined by $\bar{r}$ (cf [5]), i.e.

$$v_{ij} = \mathcal{P}_{ij}^m \left(X_i^-, \ldots, X_j^- \right) \otimes v_0 \tag{67}$$

where v_0 is the highest weight vector of $V^{\Lambda(\bar{r})}$. In particular, in the case of simple roots, i.e. when $m_i = m_{ii} = r_i + 1 \in \mathbb{N}$, we have

$$\mathcal{I}_i^{m_i} = \left(\pi_R(X_i^-) \right)^{m_i} \qquad m_i \in \mathbb{N} \tag{68a}$$

$$I_i^{m_i} = \left(\hat{\pi}_R(X_i^-) \right)^{m_i} \qquad m_i \in \mathbb{N}. \tag{68b}$$

For the non-simple roots one should use the explicit expressions for the singular vectors of the Verma modules over $U_q(sl(n))$ given in [16]. Implementing the above, one should be careful since $\hat{\pi}_R(X_i^-)$ is not preserving the reduced spaces $\mathcal{C}_{\bar{r}}, \hat{\mathcal{C}}_{\bar{r}}$, which is of course a prerequisite for (65), (66) and (68).

5. The case of $U_q(sl(3))$

In this section we consider in more detail the case $n = 3$. We could have also started (following the chronology) with the case $n = 2$ involving functions of one variable [8]. However, though by a different method, this case was obtained in [9]. It can also be obtained by restricting the construction for the (complexification of the) Lorentz quantum algebra of [7] to one of its $U_q(sl(2))$ subalgebras.

Let us now for $n = 3$ denote the coordinates on the flag manifold by $\xi = Y_{21}, \eta = Y_{32}, \zeta = Y_{31}$. We note for future use the commutation relations between these coordinates:

$$\xi\eta = q\eta\xi - \lambda\zeta \qquad \eta\zeta = q\zeta\eta \qquad \zeta\xi = q\xi\zeta. \tag{69}$$

The reduced functions for the $\mathcal{U}$ action are (cf (54)):

$$\tilde{\varphi}(\bar{Y}, \bar{D}) = \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} \xi^j \zeta^n \eta^\ell (D_1)^{r_1} (D_2)^{r_2} \tag{70a}$$

$$= \sum_{j,n,\ell \in \mathbb{Z}_+} \mu_{j,n,\ell} \tilde{\varphi}_{jn\ell} \tag{70b}$$

$$\tilde{\varphi}_{jn\ell} = \xi^j \zeta^n \eta^\ell (D_1)^{r_1} (D_2)^{r_2}. \tag{70c}$$

Now the action of $U_q(sl(3))$ on (70) is given explicitly by

$$\pi(k_1)\tilde{\varphi}_{jn\ell} = q^{j+(n-\ell-r_1)/2}\tilde{\varphi}_{jn\ell} \tag{71a}$$

$$\pi(k_2)\tilde{\varphi}_{jn\ell} = q^{\ell+(n-j-r_2)/2}\tilde{\varphi}_{jn\ell} \tag{71b}$$

$$\pi(X_1^+)\tilde{\varphi}_{jn\ell} = q^{(1+n-\ell-r_1)/2}[n+j-\ell-r_1]_q\tilde{\varphi}_{j+1,n\ell}$$
$$+ q^{j+(n-\ell-3r_1-1)/2}[\ell]_q\tilde{\varphi}_{j,n+1,\ell-1} \tag{71c}$$

$$\pi(X_2^+)\tilde{\varphi}_{jn\ell} = q^{(1+n-j-r_2)/2}[\ell-r_2]_q\tilde{\varphi}_{jn,\ell+1}$$
$$- q^{-\ell+(j-n+r_2-1)/2}[j]_q\tilde{\varphi}_{j-1,n+1,\ell} \tag{71d}$$

$$\pi(X_1^-)\tilde{\varphi}_{jn\ell} = q^{(\ell-n+r_1-1)/2}[j]_q\tilde{\varphi}_{j-1,n\ell} \tag{71e}$$

$$\pi(X_2^-)\tilde{\varphi}_{jn\ell} = -q^{(n-j+r_2-1)/2}[\ell]_q\tilde{\varphi}_{jn,\ell-1}$$
$$- q^{-\ell+(n-j+r_2-1)/2}[n]_q\tilde{\varphi}_{j+1,n-1,\ell} \,. \tag{71f}$$

It is easy to check that $\pi(k_i)$, $\pi(X_i^\pm)$ satisfy (15). It is also clear that we can remove the inessential phases by setting

$$\tilde{\pi}_{r_1,r_2}(k_i) = \pi(k_i)$$
$$\tilde{\pi}_{r_1,r_2}(X_i^\pm) = q^{\pm(r_i-1)/2}\pi(X_i^\pm). \tag{72}$$

Then $\tilde{\pi}_{r_1,r_2}$ also satisfy (15).

Now we consider the restricted functions (cf (60)):

$$\hat{\varphi}(\bar{Y}) = \sum_{j,n,\ell\in\mathbb{Z}_+} \mu_{j,n,\ell}\xi^j\zeta^n\eta^\ell \tag{73a}$$

$$= \sum_{j,n,\ell\in\mathbb{Z}_+} \mu_{j,n,\ell}\hat{\varphi}_{jn\ell} \tag{73b}$$

$$\hat{\varphi}_{jn\ell} = \xi^j\zeta^n\eta^\ell. \tag{73c}$$

As a consequence of the intertwining property (61) we obtain that $\hat{\varphi}_{jn\ell}$ obey the same transformation rules (71) as $\tilde{\varphi}_{jn\ell}$, i.e. (cf also (72)) we have

$$\pi_{r_1,r_2}(k_1)\hat{\varphi}_{jn\ell} = q^{j+(n-\ell-r_1)/2}\hat{\varphi}_{jn\ell} \tag{74a}$$

$$\hat{\pi}_{r_1,r_2}(k_2)\hat{\varphi}_{jn\ell} = q^{\ell+(n-j-r_2)/2}\hat{\varphi}_{jn\ell} \tag{74b}$$

$$\hat{\pi}_{r_1,r_2}(X_1^+)\hat{\varphi}_{jn\ell} = q^{(n-\ell)/2}[n+j-\ell-r_1]_q\hat{\varphi}_{j+1,n\ell}$$
$$+ q^{j-r_1-1+(n-\ell)/2}[\ell]_q\hat{\varphi}_{j,n+1,\ell-1} \tag{74c}$$

$$\hat{\pi}_{r_1,r_2}(X_2^+)\hat{\varphi}_{jn\ell} = q^{(n-j)/2}[\ell-r_2]_q\hat{\varphi}_{jn,\ell+1}$$
$$- q^{r_2-1-\ell+(j-n)/2}[j]_q\hat{\varphi}_{j-1,n+1,\ell} \tag{74d}$$

$$\hat{\pi}_{r_1,r_2}(X_1^-)\hat{\varphi}_{jn\ell} = q^{(\ell-n)/2}[j]_q\hat{\varphi}_{j-1,n\ell} \tag{74e}$$

$$\hat{\pi}_{r_1,r_2}(X_2^-)\hat{\varphi}_{jn\ell} = -q^{(n-j)/2}[\ell]_q\hat{\varphi}_{jn,\ell-1}$$
$$- q^{-\ell+(n-j)/2}[n]_q\hat{\varphi}_{j+1,n-1,\ell} \,. \tag{74f}$$

Let us introduce the following operators acting on our functions:

$$\hat{M}_\kappa^\pm\hat{\varphi}(\bar{Y}) = \sum_{j,n,\ell\in\mathbb{Z}_+} \mu_{j,n,\ell}\hat{M}_\kappa^\pm\hat{\varphi}_{jn\ell} \tag{75a}$$

$$T_\kappa\hat{\varphi}(\bar{Y}) = \sum_{j,n,\ell\in\mathbb{Z}_+} \mu_{j,n,\ell}T_\kappa\hat{\varphi}_{jn\ell} \tag{75b}$$

where $\kappa = \xi, \eta, \zeta$, and the explicit action on $\hat{\varphi}_{jn\ell}$ is defined by

$$\hat{M}_\xi^\pm \hat{\varphi}_{jn\ell} = \hat{\varphi}_{j\pm 1,n\ell} \tag{76a}$$

$$\hat{M}_\eta^\pm \hat{\varphi}_{jn\ell} = \hat{\varphi}_{jn,\ell\pm 1} \tag{76b}$$

$$\hat{M}_\zeta^\pm \hat{\varphi}_{jn\ell} = \hat{\varphi}_{j,n\pm 1,\ell} \tag{76c}$$

$$T_\xi \hat{\varphi}_{jn\ell} = q^j \hat{\varphi}_{jn\ell} \tag{76d}$$

$$T_\eta \hat{\varphi}_{jn\ell} = q^\ell \hat{\varphi}_{jn\ell} \tag{76e}$$

$$T_\zeta \hat{\varphi}_{jn\ell} = q^n \hat{\varphi}_{jn\ell}. \tag{76f}$$

Now we define the q-difference operators by

$$\hat{\mathcal{D}}_\kappa \hat{\varphi}(\bar{Y}) = \frac{1}{\lambda} \hat{M}_\kappa^- (T_\kappa - T_\kappa^{-1}) \hat{\varphi}(\bar{Y}) \qquad \kappa = \xi, \eta, \zeta. \tag{77}$$

Thus, we have

$$\hat{\mathcal{D}}_\xi \hat{\varphi}_{jn\ell} = [j] \hat{\varphi}_{j-1,n\ell} \tag{78a}$$

$$\hat{\mathcal{D}}_\eta \hat{\varphi}_{jn\ell} = [\ell] \hat{\varphi}_{jn,\ell-1} \tag{78b}$$

$$\hat{\mathcal{D}}_\zeta \hat{\varphi}_{jn\ell} = [n] \hat{\varphi}_{j,n-1,\ell}. \tag{78c}$$

Of course, for $q \to 1$ we have $\hat{\mathcal{D}}_\kappa \to \partial_\kappa \equiv \partial/\partial\kappa$.

In terms of the above operators the transformation rules (74) are written as

$$\hat{\pi}_{r_1,r_2}(k_1)\hat{\varphi}(\bar{Y}) = q^{-r_1/2} T_\xi T_\zeta^{1/2} T_\eta^{-1/2} \hat{\varphi}(\bar{Y}) \tag{79a}$$

$$\hat{\pi}_{r_1,r_2}(k_2)\hat{\varphi}(\bar{Y}) = q^{-r_2/2} T_\eta T_\zeta^{1/2} T_\xi^{-1/2} \hat{\varphi}(\bar{Y}) \tag{79b}$$

$$\hat{\pi}_{r_1,r_2}(X_1^+)\hat{\varphi}(\bar{Y}) = (1/\lambda)\hat{M}_\xi T_\zeta^{1/2} T_\eta^{-1/2} \left(q^{-r_1} T_\xi T_\zeta T_\eta^{-1} - q^{r_1} T_\xi^{-1} T_\zeta^{-1} T_\eta \right) \hat{\varphi}(\bar{Y})$$
$$+ q^{-r_1-1}\hat{M}_\zeta \hat{\mathcal{D}}_\eta T_\xi T_\zeta^{1/2} T_\eta^{-1/2} \hat{\varphi}(\bar{Y}) \tag{79c}$$

$$\hat{\pi}_{r_1,r_2}(X_2^+)\hat{\varphi}(\bar{Y}) = (1/\lambda)\hat{M}_\eta T_\zeta^{1/2} T_\xi^{-1/2} \left(q^{-r_2} T_\eta - q^{r_2} T_\eta^{-1} \right) \hat{\varphi}(\bar{Y})$$
$$- q^{r_2-1}\hat{M}_\zeta \hat{\mathcal{D}}_\xi T_\xi^{1/2} T_\zeta^{-1/2} T_\eta^{-1} \hat{\varphi}(\bar{Y}) \tag{79d}$$

$$\hat{\pi}_{r_1,r_2}(X_1^-)\hat{\varphi}(\bar{Y}) = \hat{\mathcal{D}}_\xi T_\zeta^{-1/2} T_\eta^{1/2} \hat{\varphi}(\bar{Y}) \tag{79e}$$

$$\hat{\pi}_{r_1,r_2}(X_2^-)\hat{\varphi}(\bar{Y}) = -\hat{\mathcal{D}}_\eta T_\zeta^{1/2} T_\xi^{-1/2} \hat{\varphi}(\bar{Y}) - \hat{M}_\xi \hat{\mathcal{D}}_\zeta T_\xi^{-1/2} T_\zeta^{1/2} T_\eta^{-1} \hat{\varphi}(\bar{Y}) \tag{79f}$$

where $\hat{M}_\kappa = \hat{M}_\kappa^+$.

Note that it is possible to obtain a realization of the representation $\hat{\pi}_{r_1,r_2}$ on monomials in three commuting variables x, y, z. Indeed, one can relate the non-commuting algebra $\mathbb{C}[\xi, \eta, \zeta]$ with the commuting algebra $\mathbb{C}[x, y, z]$ by fixing an ordering prescription. However, such realization in commuting variables may be obtained much more directly as is done by other methods and for other purposes in [17]. In the present paper we are interested in the non-commutative case and we continue to work with the non-commuting variables ξ, η, ζ.

Now we can illustrate some of the general statements of the previous section. Let $m_2 = r_2 + 1 \in \mathbb{N}$. It is clear that if $\ell \geqslant m_2$, functions $\hat{\varphi}$ from (73) with $\mu_{j,n,\ell} = 0$ form an invariant subspace, since

$$\hat{\pi}_{r_1,r_2}(X_2^+)\hat{\varphi}_{jnr_2} = -q^{-1+(j-n)/2}[j]_q \hat{\varphi}_{j-1,n+1,r_2} \tag{80}$$

and all other operators in (74) either preserve or lower the index ℓ. The same is true for the functions $\tilde{\varphi}$. In particular, for $m_2 = 1$ the functions in the invariant subspace do not depend on the variable η. In this case we have functions of two q-commuting variables $\zeta\xi = q\xi\zeta$ which are much easier to handle that the general non-commutative case (69).

The intertwining operator (68) for $m_2 \in \mathbb{N}$ is given as follows. First we calculate

$$
\begin{aligned}
\left(\pi_R(X_2^-)\right)^s \tilde{\varphi}_{jn\ell} &= \left(\pi_R(X_2^-)\right)^s \xi^j \zeta^n \eta^\ell D_1^{r_1} D_2^{r_2} \\
&= \xi^j \zeta^n \sum_{t=0}^{s} a_{st} \eta^{\ell-t} D_1^{r_1+t} D_2^{r_2-s-t} (\xi_{13}^{12})^{s-t}
\end{aligned}
\tag{81}
$$

$$
a_{st} = q^{t\ell+r_2 s/2-(s+t)(s+t+1)/4} \binom{s}{t}_q \frac{[r_2-t]_q![\ell]_q!}{[r_2-s]_q![\ell-t]_q!}
$$

where $\binom{n}{k}_q \equiv [n]_q!/[k]_q![n-k]_q!$, $[m]_q! \equiv [m]_q[m-1]_q\ldots[1]_q$. Thus, $\pi_R(X_2^-)$ is not preserving the reduced space $\mathcal{C}_{r_1,r_2}$, and furthermore there is the additional variable ξ_{13}^{12}. Since we would like $\pi_R(X_2^-)$ to some power to map to another reduced space, this is only possible if the coefficients a_{st} vanish for $s \neq t$. This happens iff $s = r_2 + 1 = m_2$. Thus we have (in terms of the representation parameters $m_i = r_i + 1$)

$$
\begin{aligned}
\left(\pi_R(X_2^-)\right)^{m_2} \xi^j \zeta^n \eta^\ell D_1^{m_1-1} D_2^{m_2-1} \\
= q^{m_2(\ell-1-m_2/2)} \frac{[\ell]_q!}{[\ell-m_2]_q!} \xi^j \zeta^n \eta^{\ell-m_2} D_1^{m_{12}-1} D_2^{-m_2-1}.
\end{aligned}
\tag{82}
$$

Comparing the powers of D_i, we recover at once (63) for our situation, namely, $m_1' = m_{12}$ and $m_2' = -m_2$. We have thus shown (64a) and (65a). Then (64b) and (65b) follow using (61). This intertwining operator has a kernel which is just the invariant subspace discussed above—it is obvious from the factor $1/[\ell-m_2]_q!$ in (82) that all monomials with $\ell < m_2$ are mapped to zero.

For the restricted functions we have

$$
\begin{aligned}
\left(\pi_R(X_2^-)\right)^{m_2} \hat{\varphi}_{jn\ell} &= q^{m_2(\ell-1-m_2/2)} \frac{[\ell]_q!}{[\ell-m_2]_q!} \hat{\varphi}_{jn,\ell-m_2} \\
&= q^{-3m_2/2} \left(\hat{\mathcal{D}}_\eta T_\eta\right)^{m_2} \hat{\varphi}_{jn\ell}.
\end{aligned}
\tag{83}
$$

Thus, renormalizing (68b) by $q^{-3m_2/2}$ we finally have

$$
I_2^{m_2} = \left(\hat{\mathcal{D}}_\eta T_\eta\right)^{m_2}.
\tag{84}
$$

For $q = 1$ this operator reduces to the known result: $I_2^{m_2} = (\partial_\eta)^{m_2}$ [5].

Let now $m_1 \in \mathbb{N}$. In a similar way, though the calculations are more complicated, we find

$$
\begin{aligned}
\left(\pi_R(X_1^-)\right)^{m_1} \xi^j \zeta^n \eta^\ell D_1^{m_1-1} D_2^{m_2-1} \\
= q^{m_1(j+n-\ell-1-m_1/2)} \sum_{t=0}^{m_1} q^{-t(t+3+2j)/2} \\
\times \binom{m_1}{t}_q \frac{[j]_q![n]_q!}{[j-m_1+t]_q![n-t]_q!} \xi^{j+t-m_1} \zeta^{n-t} \eta^{\ell+t} D_1^{-m_1-1} D_2^{m_{12}-1}.
\end{aligned}
\tag{85}
$$

Comparing the powers of D_i we recover (63) for our situation, namely, $m_1' = -m_1$, $m_2' = m_{12}$. Thus, we have shown that (64) and (65) hold.

For the restricted functions we have

$$
\left(\pi_R(X_1^-)\right)^{m_1} \hat{\varphi}_{jn\ell} = q^{m_1(j+n-\ell-1-m_1/2)} \sum_{t=0}^{m_1} q^{t(t+3+2j)/2}
$$

$$
\times \binom{m_1}{t}_q \frac{[j]_q![n]_q!}{[j-m_1+t]_q![n-t]_q!} \hat{\varphi}_{j+t-m_1,n-t,\ell+t}
$$

$$
= q^{-m_1(3/2+m_1)} T_\zeta^{m_1} \sum_{t=0}^{m_1} \hat{M}_\eta^t \hat{\mathcal{D}}_\zeta^t (q\hat{\mathcal{D}}_\xi T_x)^{m_1-t} T_\eta^{-m_1} \hat{\varphi}_{jn\ell}. \tag{86}
$$

Then, renormalizing (68b) we finally have

$$
I_1^{m_1} = T_\zeta^{m_1} \sum_{t=0}^{m_1} \hat{M}_\eta^t \hat{\mathcal{D}}_\zeta^t (q\hat{\mathcal{D}}_\xi T_x)^{m_1-t} T_\eta^{-m_1}. \tag{87}
$$

For $q = 1$ this operator reduces to a known result: $I_1^{m_1} = (\partial_\xi + \eta \partial_\zeta)^{m_1}$ [5].

Finally, let us consider the case $m = m_{12} = m_1 + m_2 \in \mathbb{N}$, first with $m_1, m_2 \notin \mathbb{N}$. In this case the intertwining operator is given by (66) and (67) with [18], formula (27) (cf also [16])

$$
\mathcal{P}_{12}^m \left(X_1^-, X_2^-\right) = \sum_{s=0}^m a_s (X_1^-)^{m-s} (X_2^-)^m (X_1^-)^s
$$

$$
a_s = (-1)^s a \frac{[m_1]_q}{[m_1 - s]_q} \binom{m}{s}_q \qquad s = 0, \ldots, m, a \neq 0. \tag{88}
$$

Let us illustrate the resulting intertwining operator in the case $m = 1$. We then have, setting in (88) $a = [1 - m_1]_q$,

$$
\mathcal{I}_{12}^1 = [1 - m_1]_q \pi_R(X_1^-) \pi_R(X_2^-) + [m_1]_q \pi_R(X_2^-) \pi_R(X_1^-). \tag{89}
$$

Then we can see at once the intertwining properties of $\mathcal{I}_{12}^1$ by calculating

$$
\mathcal{I}_{12}^1 \xi^j \zeta^n \eta^\ell D_1^{m_1-1} D_2^{m_2-1} = q^{j+n-2-m_1} [j]_q [\ell]_q \xi^{j-1} \zeta^n \eta^{\ell-1} D_1^{m_1-2} D_2^{m_2-2}
$$

$$
+ q^{n-2} [n]_q [\ell + m_1]_q \xi^j \zeta^{n-1} \eta^\ell D_1^{m_1-2} D_2^{m_2-2}. \tag{90}
$$

Comparing the powers of D_i we recover (63) for our situation, namely, $m_1' = -m_2 = m_1 - 1$, $m_2' = -m_1 = m_2 - 1$.

For the restricted functions we have

$$
\left([1 - m_1]_q \pi_R(X_1^-) \pi_R(X_2^-) + [m_1]_q \pi_R(X_2^-) \pi_R(X_1^-)\right) \hat{\varphi}_{jn\ell}
$$

$$
= q^{n-2+j-m_1} [j]_q [\ell]_q \hat{\varphi}_{j-1,n,\ell-1} + q^{n-2} [n]_q [\ell + m_1]_q \hat{\varphi}_{j,n-1,\ell}
$$

$$
= q^{-2} \left(q^{-m_1} \hat{\mathcal{D}}_\xi T_\xi \hat{\mathcal{D}}_\eta + (1/\lambda) \hat{\mathcal{D}}_\zeta (q^{m_1} T_\eta - q^{-m_1} T_\eta^{-1}) \right) T_\zeta \hat{\varphi}_{jn\ell}. \tag{91}
$$

Rescaling (66b) we finally have

$$
I_{12}^1 = \left(q^{-m_1} \hat{\mathcal{D}}_\xi T_\xi \hat{\mathcal{D}}_\eta + (1/\lambda) \hat{\mathcal{D}}_\zeta (q^{m_1} T_\eta - q^{-m_1} T_\eta^{-1}) \right) T_\zeta. \tag{92}
$$

For $q = 1$ this operator is $I_{12}^{m=1} = \partial_\xi \partial_\eta + (m_1 + \eta \partial_\eta) \partial_\zeta$ [5].

Above we have assumed that $m_1, m_2 \notin \mathbb{N}$. However, after the proper choice of a in (88), (e.g. as made above in (89)) we can consider the singular vector (88) and the resulting intertwining operator also when m_1 and/or m_2 are positive integers. Of particular interest are the cases $m_1, m_2 \in \mathbb{Z}_+$. In these cases the singular vector is reduced in four different ways (cf [16, formulae (33a-d)], [18]). Accordingly, the intertwining operator becomes composite, i.e. it can be expressed as a composition of the intertwiners introduced so far as follows

$$I_{12}^m = c_1 I_1^{m_2} I_2^m I_1^{m_1} \tag{93a}$$
$$= c_2 I_2^{m_1} I_1^m I_2^{m_2} \tag{93b}$$
$$= c_3 I_2^{m_1} I_{12}^{m_2} I_1^{m_1} \tag{93c}$$
$$= c_4 I_1^{m_2} I_{12}^{m_1} I_2^{m_2}. \tag{93d}$$

The four expressions were used to prove commutativity of the hexagon diagram of $U_q(sl(3, \mathbb{C}))$ [18]. This diagram involves six representations which are denoted by V_{00}, V_{00}^1, V_{00}^2, V_{00}^{12}, V_{00}^{21} and V_{00}^3 in (29) of [18] and which in our notation are connected by the intertwiners in (93) as follows:

$$\hat{C}_{m_1,m_2} \xrightarrow{I_1^{m_1}} \hat{C}_{-m_1,m} \xrightarrow{I_2^m} \hat{C}_{m_2,-m} \xrightarrow{I_1^{m_2}} \hat{C}_{-m_2,-m_1} \tag{94a}$$

$$\hat{C}_{m_1,m_2} \xrightarrow{I_2^{m_2}} \hat{C}_{m,-m_2} \xrightarrow{I_1^m} \hat{C}_{-m,m_1} \xrightarrow{I_2^{m_1}} \hat{C}_{-m_2,-m_1} \tag{94b}$$

$$\hat{C}_{m_1,m_2} \xrightarrow{I_1^{m_1}} \hat{C}_{-m_1,m} \xrightarrow{I_{12}^{m_2}} \hat{C}_{-m,m_1} \xrightarrow{I_2^{m_1}} \hat{C}_{-m_2,-m_1} \tag{94c}$$

$$\hat{C}_{m_1,m_2} \xrightarrow{I_2^{m_2}} \hat{C}_{m,-m_2} \xrightarrow{I_{12}^{m_1}} \hat{C}_{m_2,-m} \xrightarrow{I_1^{m_2}} \hat{C}_{-m_2,-m_1}. \tag{94d}$$

Of these six representations only $\hat{C}_{m_1,m_2}$ has a finite dimensional irreducible subspace iff $m_1 m_2 > 0$, the dimension being $m_1 m_2 m/2$ [18]. If $m_1 = 0$, the intertwining operators with superscript m_1 become the identity (since in these cases the intertwined spaces coincide) and the compositions in (93), (94) are shortened to two terms in the cases (a,b,d) and to one term in the case (c), (respectively, for $m_2 = 0$, two terms in the cases (a,b,c), one term in (d)). Such considerations are part of the multiplet classification given in [18].

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